1) $\vec{F} = \langle -xy^2, x^2y, e^{5z} \rangle$. (Note: this problem takes place in $\mathbb{R}^3$)

a) Show that $\nabla \times \vec{F}$ is tangent to the cylinder

$$x^2 + y^2 = 1.$$

b) Compute

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C is $\vec{r}(t) = \langle \cos t, \sin t, \sin^3(t)\cos^4(t) \rangle$

$$0 \le t \le 2\pi.$$

Hint: a) To show tangency, show dot prod. w/ normal to surface is zero.

2) Let S be the portion of the elliptic paraboloid

$$z = x^2 + 4y^2 - 4$$

that is underneath the xy-plane, with the downwards orientation. Let

$$\vec{F} = \langle y \log_2(x^2+4y^2+z^2) + 3x^2y^2\cos(x^3),$$

$$-3x + 2y\sin(x^3),$$

$$e^{yz}\arctan(y^{x^2+1})\rangle.$$

Compute $\displaystyle\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

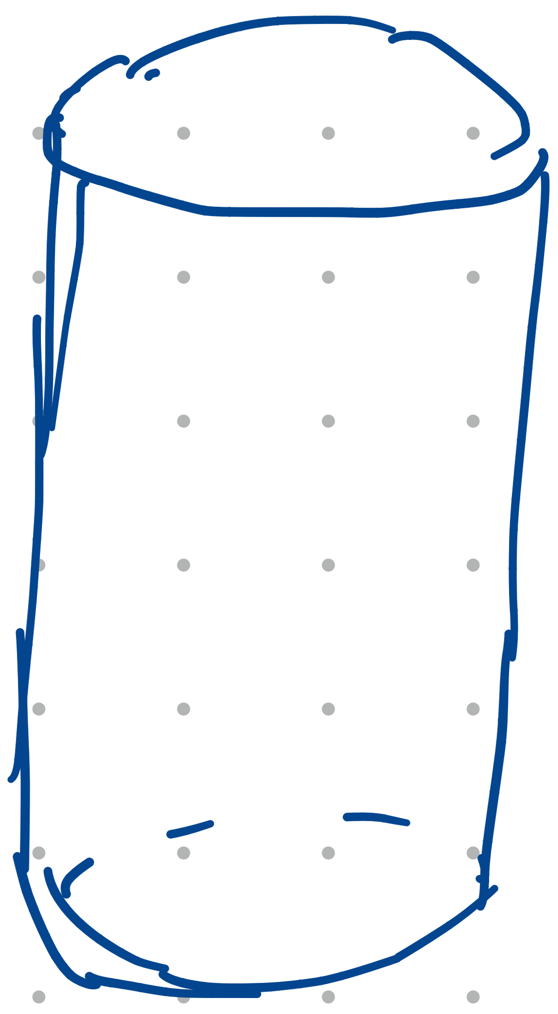
Hints: pretty clear indicator that Stokes is involved.

Solution to 1)

a) $\nabla \times \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -xy^2 & x^2y & e^{5z} \end{bmatrix}$

$$= \langle 0-0,\ 0-0,\ 2xy-(-2xy) \rangle$$

$$= \langle 0, 0, 4xy \rangle.$$



(At this point, it may be geometrically clear that (a) is true, but let's verify algebraically too)

A normal vec for cylinder:

$$\langle 2x, 2y, 0 \rangle = \nabla(x^2+y^2-1)$$

(this is not a unit normal, but that's totally fine for this problem.)

indeed $\langle 0,0,4xy \rangle \cdot \langle 2x, 2y, 0 \rangle = 0$.

Q: Is the vector field $\langle 2x, 2y - \frac{2}{y}, 4xy \rangle$

tangent to the cylinder (at all points where this makes sense)?

A: check:

$\langle 2x, 2y - \frac{2}{y}, 4xy \rangle \cdot \langle 2x, 2y, 0 \rangle$

$= 4x^2 + 4y^2 - 4 + 0$

$= 0$ on the cylinder. ✓

2Dex):

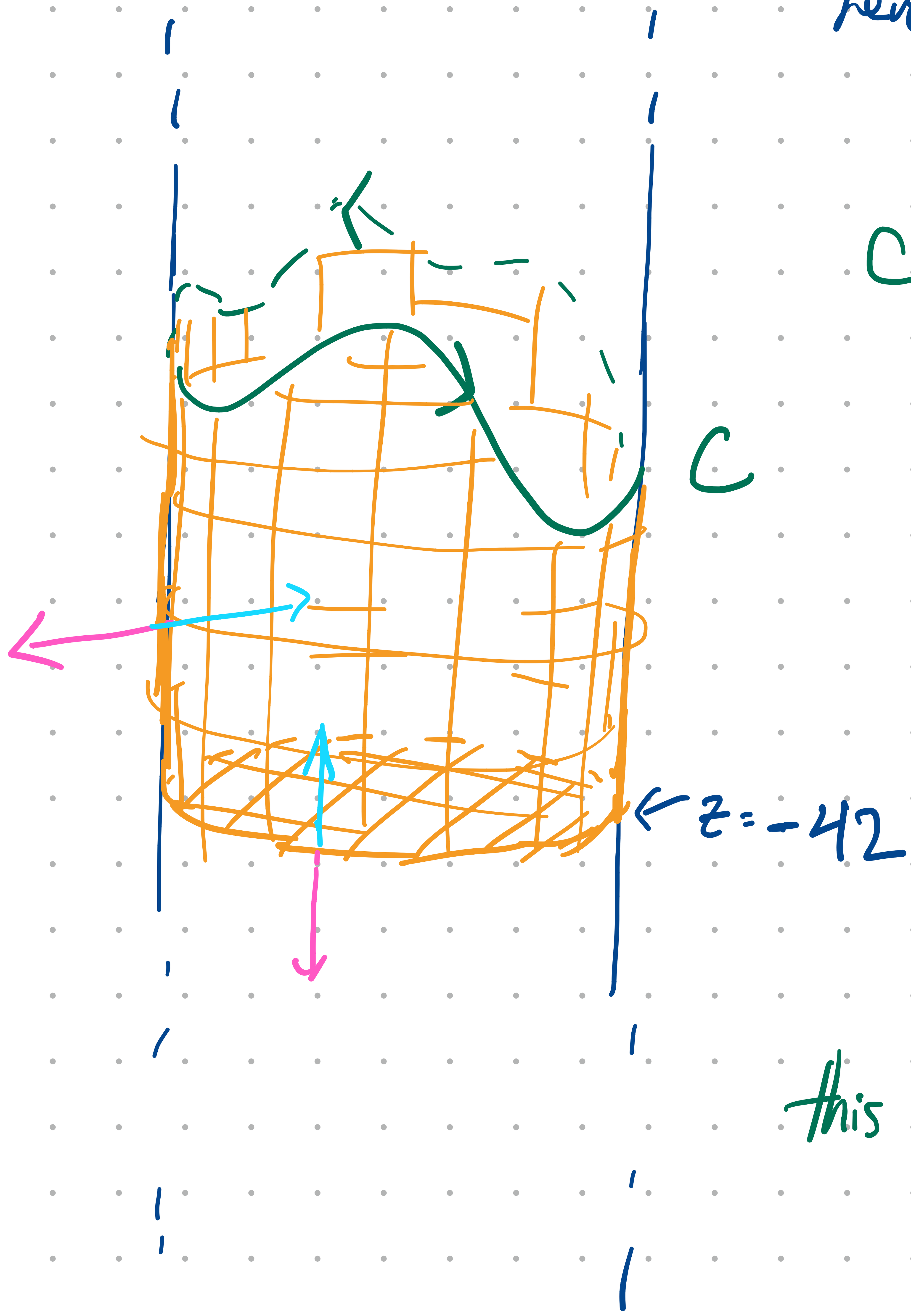
$x^2 - y^2 = 1$ hyperbola $\langle 2x, -2y \rangle$

Q: is $\langle x, y + \frac{1}{y} \rangle$ tangent to hyperbola?

$\langle x, y + \frac{1}{y} \rangle \cdot \langle 2x, -2y \rangle = 2x^2 - 2y^2 - 2$

$= 0.$ ✓

b) Suspicion from (a) that Stokes could be useful here...



C resides on the cylinder b/c

$$(\cos t)^2 + (\sin t)^2 = 1$$

$$\underset{x}{\cos t} \qquad \underset{y}{\sin t}$$

from parametrization

this is true for <u>all</u> t.

RHR says the orientation on  compatible with C is the cyan one.

Stokes:

$$\int_C \vec{F}\cdot d\vec{r} = \iint (\nabla\times\vec{F})\cdot d\vec{S}$$

$$+ \iint (\nabla\times\vec{F})\cdot d\vec{S}$$

$$= \iint \underbrace{\langle 0,0,4xy\rangle\cdot d\vec{S}}_{=0 \text{ by (a)}}$$

$$+ \iint \langle 0,0,4xy\rangle\cdot d\vec{S}$$

$$= \iint_{x^2+y^2\le 1} 4xy\,dx\,dy$$ = proceed w/ Ch. 15 methods

or argue by symmetry

$$= \boxed{0}$$

Math 53: Multivariable Calculus	**Instructor: Maciej R. Zworski // GSI: Xianglong Ni**

Quiz 12: Stokes' Theorem and Applications

Your name: *Discussion 214 // 2019-05-01*

Instructions. Pick TWO of the following four questions. Clearly INDICATE which two problems you would like graded (the other two will not be graded).

CIRCLE THE TWO PROBLEMS YOU WANT GRADED: **1** **2** **3** **4**

Problem 1 (6 points)**.** Let S be the surface

$$z = x^2 + 4y^2 - 4, \qquad z \leq 0$$

oriented downwards (i.e. negatively). Compute $\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$, where

$$\mathbf{F} = \big(y \log_2(x^2 + 4y^2 + z^2) + 3x^2 y^2 \cos(x^3)\big)\hat{\mathbf{i}} + \big(-3x + 2y\sin(x^3)\big)\hat{\mathbf{j}} + \big(e^{yz} \arctan(y^{x^2+1})\big)\hat{\mathbf{k}}$$

Solution: The oriented boundary of the oriented surface S is the ellipse

$$z = 0, \qquad 0 = x^2 + 4y^2 - 4$$

in the xy-plane, with the "negative" orientation—which is to say, clockwise when viewed from above. Let C denote this oriented curve. By Stokes' theorem, we have that

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

We can expand the right hand side of this equation as

$$\int_C \big(y \log_2(x^2 + 4y^2 + z^2) + 3x^2 y^2 \cos(x^3)\big)\,\mathrm{d}x + \big(-3x + 2y\sin(x^3)\big)\,\mathrm{d}y + \big(e^{yz}\arctan(y^{x^2+1})\big)\,\mathrm{d}z.$$

However, since the curve C is contained in the horizontal plane $z = 0$, we have that $\mathrm{d}z = 0$ (the z-coordinate is unchanging!). Moreover, we can use the equations of the curve C to simplify the integrand:

$$\int_C \big(y \log_2(4 + 0^2) + 3x^2 y^2 \cos(x^3)\big)\,\mathrm{d}x + \big(-3x + 2y\sin(x^3)\big)\,\mathrm{d}y$$
$$= \int_C \big(2y + 3x^2 y^2 \cos(x^3)\big)\,\mathrm{d}x + \big(-3x + 2y\sin(x^3)\big)\,\mathrm{d}y.$$

Now the problem looks very similar to Midterm 2 Problem 2. We can apply Green's theorem, but remember that C is oriented negatively and thus we get a minus sign:

$$-\iint_D \big((-3 + 6x^2 y\cos(x^3)) - (2 + 6x^2 y \cos(x^3))\big)\,\mathrm{d}A = -\iint_D -5\,\mathrm{d}A = 5\,\mathrm{Area}(D)$$

where D is the region enclosed by the ellipse C. So the answer is $\boxed{10\pi}$. □

Solution 2: A completely equivalent method is to use Stokes' theorem again instead of Green's theorem (the latter is just a special case of the former anyway). The oriented curve C bounds the solid ellipse D oriented negatively (downwards). The vector surface element $\mathrm{d}\mathbf{S}$ on D is given by $\langle 0, 0, -1\rangle\,\mathrm{d}x\,\mathrm{d}y$. Applying Stokes' to this region and the simplified vector field gives the same answer as the preceding method. □

(Continued on back.)

Hints: Does the vector field $\mathbf{F}$ remind you of one you've encountered before? Similar principles may be applicable.